Interpolation (For equal Interval)

The process of computing intermediate values of a function from a given set of tabular values of the function is called Interpolation.

Suppose ~~the~~ a set of corresponding values of x & y,

$x: \quad x_0 \quad x_1 \quad x_2 \dots x_n$

$y: \quad y_0 \quad y_1 \quad y_2 \dots y_n$

Let $y_i = f(x_i)$, $i = 0, 1, 2, \dots, n$. If $f(x)$ is known, the value of y can be calculated for any x. But in many cases, to find $y = f(x)$ such that $y_i = f(x_i)$, from the above. This is not easy because there are infinity of functions $y = \phi(x)$ such that $y_i = \phi(x_i)$. Of the sequence of functions $\{\phi(x)\}$, there is a unique n^{th} degree polynomial $P_n(x)$ such that $y_i = P_n(x_i)$, $i = 0, 1, 2, \dots, n$.

The function $\phi(x)$ is called interpolating function. This polynomial function $P_n(x)$ may be taken as an interpolating polynomial where

$$y_i = f(x_i) = P_n(x_i), \quad i = 0, 1, 2, \dots, n.$$

Interpolation Methods:

The methods of interpolation are

(i) Newton Forward formula.

(ii) Newton Backward formula.

(iii) Binomial Method and
(iv) Lagrange's Method. $\Big|\rightarrow$ not included in Sylabus.

(i) Newton's Forward Formula:

Let $x_0, x_1, x_2, \ldots, x_n$ be the set of equidistant values of the variable x.

$\therefore \; x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = h$ (say).

Let $u = \dfrac{x - x_0}{h}$.

Formula: Newton Forward Difference

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$
$$+ \frac{u(u-1)(u-2)+\cdots+u(u-1)(u-2)\ldots(u-(n-1))}{n!}\cdot \Delta^n y_0.$$

Pf.

$$P_n(x) = P_n(x_0 + uh)$$
$$= E^u P_n(x_0).$$
$$= E^u(y_0)$$
$$= (1+\Delta)^u y_0$$
$$= \{1 + {}^uc_1 \Delta + {}^uc_2 \Delta^2 + \cdots + {}^uc_n \Delta^n + \cdots\} y_0$$
$$= y_0 + {}^uc_1 \Delta y_0 + {}^uc_2 \Delta^2 y_0 + \cdots + {}^uc_n \Delta^n y_0 + \cdots$$

where $u = \dfrac{x - x_0}{h}$.

If $y(x)$ is a polynomial of n^{th} degree, $\Delta^{n+1} y_0, \ldots$ are zero.

Hence $P_n(x) = P_n(x_0 + uh) = y_0 + {}^uc_1 \Delta y_0 + {}^uc_2 \Delta^2 y_0 + \cdots + {}^uc_n \Delta^n y_0.$

Note:

(i) The first two terms will give the linear interpolation and the first three terms will give the parabolic interpolation and so on.

(ii) Since the formula involves the forward differences of y_0, we call it as Newton forward interpolation formula.

(iii) This is applicable only if the interval of differencing h is a constant.

(ii) Gregory - Newton Backward Interpolation Formula.

Newton's forward interpolation formula cannot be used for interpolating a value of y nearer to the end of the table values. For this purpose, we get another backward interpolation formula.

Let $x_0, x_1, x_2, \dots, x_n$ be a set of equidistant values of the arguments x and $y_0, y_1, y_2, \dots, y_n$ be the values of the function $y = f(x)$.

Let $x_0 - x_1 = x_2 - x_1 = \dots = x_n - x_{n-1} = h$ (say) and

let $V = \frac{x - x_n}{h}$.

Then the Newtons backward difference formula is

$$y = y_n + \frac{v}{1!} \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n + \dots$$

Pf:

$$P_n(x) = P_n(x_n + vh) \quad , \text{where} \quad v = \frac{x - x_n}{h}$$

$$= E^v P_n(x_n)$$

$$= (1-\nabla)^{-v} y_n \qquad \left\{ \because E = (1-\nabla)^{-1} \right\}$$

$$= \left\{ 1 + v\nabla + \frac{v(v+1)}{2!}\nabla^2 + \frac{v(v+1)(v+2)}{3!}\nabla^3 + \dots \right\} y_n$$

$$\boxed{P_n(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots}$$

Problem

1. Find the values of y at $x = 21$ and $x = 28$ from the following table.

x :	20	23	26	29
y :	0.3420	0.3907	0.4384	0.4848

Sol

Since $x = 21$ is nearer to the begining of the table, we use Newton's forward formula. (NFF)

We form the difference table with $h = 3$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.3420			
		0.0487		
23	0.3907		-0.0010	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

The top most diagonal gives the NFF of y_0 while the lowermost diagonal gives the backward differences of y_n

There are only 4 data given. Hence the collocation polynomial will be of degree 3.

By Newton Forward Interpolation formula,

$$y(x) = P_3(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 ;$$

where $u = \frac{x-x_0}{h} = \frac{21-20}{3} = 0.3333.$

$$y(21) = P_3(21) = 0.3420 + 0.3333(0.0487) + \frac{0.3333(-0.6666)}{2!}(-0.0010)$$

$$+ \frac{0.3333(-0.6666)(-1.6666)}{6}(0.0003)$$

$$\therefore \boxed{y(21) = 0.3583}$$

Since $x = 28$ is nearer to end value, we use Newton's backward interpolation formula.

$$y(x) = P_3(x) = P_3(x_n + vh)$$

$$= y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{6}\nabla^3 y_n + \cdots$$

$$y(28) \simeq P_3(28) = P_3\left[29 + \left(-\frac{1}{3}\right)3\right]$$

where

$$v = \frac{x-x_n}{h} = \frac{28-29}{3} = -\frac{1}{3}.$$

$$y(28) = 0.4848 + \left(-\frac{1}{3}\right)0.0464 + \frac{\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)}{2}(-0.0013) + \frac{\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{6}(-0.0002) + \cdots$$

$$= 0.4848 - 0.015465 + 0.0001444 + 0.0000185$$

$$\boxed{y(28) = 0.4695}$$

Example (1) From the following data find y at $x = 43$ and $x = 84$.

x:	40	50	60	70	80	90
y:	184	204	226	250	276	304.

(2)

x:	0	1	2	3
$f(x)$:	11	7	6	10

, find $f(2.5)$ using Newton's backward formula.

Equidistant terms with one or more missing values:

When one or more of the values of the function $y = f(x)$ corresponding to the equidistant values of x are missing. We can find the missing values by the use of operators Δ and E.

Ex: 1 Find the missing values of the table given below. What assumption have you made to find it?

Year :	1917	18	19	20	21
Export : (in tons)	443	384	-	397	467.

Sol Since 4 values are given, we make an assumption that we get a third degree polynomial.

Hence the 4^{th} differences of $P_3(x)$ are zeros.

We can, without any loss, assume

$$u_0 = 443, \; u_1 = 384, \; u_2 = ?, \; u_3 = 397, \; u_4 = 467$$

$$\Delta^4 u_0 = 0$$

$$(E-1)^4 u_0 = 0.$$

i.e., $(E^4 - 4E^3 + 6E^2 - 4E + 1)u_0 = 0$

$$u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0$$

$$\therefore \boxed{u_2 = 369}$$

Ex-2 Find the missing value of the following table.

x :	0	1	2	3	4
y :	1	2	4	-	16.

Sol Taking $y_x = P_3(x)$, $y_0 = 1$, $y_1 = 2$, $y_2 = 4$, $y_3 = ?$, $y_4 = 16$.

$$\therefore \boxed{y_3 = 8.25}$$